arXiv:2610.35651v3 [math.DG] 8 Nov 2026

INDEX AND NULLITY OF CRITICAL SPHERICAL CATENOIDS IN HYPERBOLIC THREE-SPACE

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Abstract. We prove that every critical spherical catenoid in a geodesic ball of hyperbolic three-space has Morse index four and nullity two. This resolves Medvedev's index conjecture throughout the parameter range of the catenoid family. The kernel consists precisely of the normal components of rotations that preserve the center of the ball and tilt the axis of the catenoid. As consequences, the radius of the supporting ball increases strictly along the family, and each critical spherical catenoid admits a smooth family of free boundary constant mean curvature deformations in its supporting ball. The proof combines a global boundary inequality obtained from a Wronskian, a positive Dirichlet eigenfunction supplied by the time component of the unit normal, and a Riccati comparison for the angular mode $k = 2$. A ground-state identity then yields positivity of every angular mode $k \geq 2$. We also show that the energy index of every critical spherical catenoid in $\mathbb{H}^3$, that is, the index of the second variation of its energy, equals four; hence the inequality between the energy index and the spectral index conjectured by Medvedev for geodesic balls of $\mathbb{H}^n$ fails.

1. Introduction

The critical catenoid is a basic example in the study of the Morse index of free boundary minimal surfaces. In the Euclidean unit ball in $\mathbb{R}^3$, its index is four, as proved by Devyver [5], Smith and Zhou [16], and Tran [17]; another proof was given by Medvedev [9]. Devyver also proved that every compact orientable free boundary minimal surface in this ball, other than a flat disk, has index at least four [5]. Thus the critical catenoid attains this lower bound. Its nullity is two, and it is equivariantly nondegenerate: its Jacobi fields satisfying the free boundary condition are induced by rotations [1, Lemma 8.1]; see also [1, Remark 8.2]. Moreover, Parise and Zhu have recently shown that every properly embedded free boundary minimal annulus in this ball is congruent to the critical catenoid [12].

The analogous index problem in curved space forms concerns critical spherical catenoids. In $\mathbb{H}^3$, these are rotational minimal annuli obtained by free boundary truncation of the spherical catenoids constructed by Mori [11]. The geometry of rotational hypersurfaces in spaces of constant curvature was developed further by do Carmo and Dajczer [2], and the free boundary truncation was considered by Li and Xiong [6]. C. Lima [7] proved that an embedded free boundary minimal annulus in a geodesic ball of $\mathbb{S}^3_+$ or $\mathbb{H}^3$ that is invariant under the geodesic symmetry about the center of the ball is congruent to a critical rotational annulus; in $\mathbb{H}^3$, these are the critical spherical catenoids. Medvedev [10, Theorem 2] proved that the critical spherical catenoid in a geodesic ball of $\mathbb{S}^3_+$ has index four, and that its hyperbolic counterpart has index at least four. He conjectured equality in the hyperbolic case [10, Remark 10]. We prove this equality throughout the parameter range of the critical spherical catenoid family in $\mathbb{H}^3$ and determine the nullity.

The analysis in [15] determines the index and nullity contributions of the angular mode $|k| = 1$ for every parameter. The local result in [14] establishes index four and nullity two when the parameter is sufficiently close to its lower endpoint. The theorem below establishes both equalities for every $a > \frac{1}{2}$.

2020 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42, 58J50, 34B24.

Key words and phrases. Free boundary minimal surfaces, hyperbolic space, critical catenoid, Morse index, nullity, Robin eigenvalue problem, equivariant nondegeneracy, constant mean curvature, energy index.

Theorem 1. *For each $a > \frac{1}{2}$, let Σ_a be the critical spherical catenoid with angular period 2π, parametrized in Section 2, and let $\mathbb{B}(r(a)) \subset \mathbb{H}^3$ be its supporting geodesic ball. Then*

$$\operatorname{ind}(\Sigma_a) = 4, \qquad \operatorname{nul}(\Sigma_a) = 2.$$

The Robin kernel is generated by the normal components of the two rotations that tilt the axis of the catenoid while fixing the center of $\mathbb{B}(r(a))$.

Here the index and nullity refer to the second variation of area for normal variations whose boundary remains on the supporting sphere. In particular, the ambient ball is fixed when taking variations.

The theorem supplies the global hyperbolic counterpart of the Euclidean index-four results and identifies all infinitesimal degeneracies with rotations of the ambient ball. Together with the Euclidean results and Medvedev's spherical result, it shows that critical catenoids have index four in the Euclidean unit ball, in geodesic balls of $\mathbb{S}^3_+$, and in every geodesic ball of $\mathbb{H}^3$. At fixed ambient curvature -1, both the geometry of the annulus and its supporting radius vary with a. The associated Jacobi operator contains the curvature term $+2$, and the Robin coefficient is $\coth r(a)$; these terms are recorded in Section 3. The proof controls this family of boundary value problems over the entire interval $a > \frac{1}{2}$.

The proof has three main ingredients. A Wronskian identity gives a strict boundary inequality valid for every $a > \frac{1}{2}$. The time component of the unit normal is a positive Dirichlet eigenfunction with eigenvalue two; a codimension-two min–max argument therefore determines the radial mode $k = 0$. Finally, a comparison of logarithmic derivatives, together with an explicit integral inequality, gives a boundary inequality for the mode $k = 2$; a ground-state identity turns it into positivity of that mode, and monotonicity in k^2 completes the spectral count. The geometric and spectral identities used below are included in the proof.

Three consequences are recorded in Section 8. The radius of the supporting ball is a strictly increasing function of a (Corollary 10). Together with Lemma 11, this proves that every geodesic ball of $\mathbb{H}^3$ is the supporting ball of an isometric image of exactly one Σ_a, as stated without proof by Li and Xiong [6]. Combined with the equivariant implicit function theorem of Bettiol, Piccione and Santoro [1], the theorem shows that each Σ_a is the member with mean curvature zero of a smooth family of free boundary CMC annuli in its supporting ball (Corollary 12). The restriction to angular period 2π is essential: for $m \geq 2$, the m-fold cover of Σ_a has index at least $4m$, and its nullity exceeds two exactly when $mk_*(a)$ is an integer, for a threshold frequency $k_*(a) \in (1, 2)$ (Remark 14). This agrees with the non-rotational free boundary minimal annuli of Cerezo [3], which bifurcate from multiple covers of critical spherical catenoids.

Medvedev also compared the energy index ind_E, the index of the second variation of the energy at fixed conformal structure, with the spectral index ind_S (denoted Ind_E and Ind_S in [10]; both are recalled in Section 9). For free boundary minimal submanifolds of geodesic balls in $\mathbb{S}^n_+$ he proved $\operatorname{ind}_E \leq n \operatorname{ind}_S$, where n is the dimension of the ambient space, and he conjectured the same inequality in $\mathbb{H}^n$ [10, Proposition 11 and Remark 7]. For annuli, V. Lima's bound $\operatorname{ind} \leq \operatorname{ind}_E + 1$ [8] then gives $\operatorname{ind} \leq n \operatorname{ind}_S + 1$; in $\mathbb{S}^3_+$ this is the upper bound for the index of the critical spherical catenoid [10, Theorem 12 and Corollary 17], since $\operatorname{ind}_S = 1$ there. In $\mathbb{H}^3$ one also has $\operatorname{ind}_S(\Sigma_a) = 1$ [10, Theorem 18] (see also Section 9), so the conjectured inequality would give $\operatorname{ind}(\Sigma_a) \leq 4$ in the same way. Our second result shows that the conjectured inequality fails in $\mathbb{H}^3$.

Theorem 2. *For every $a > \frac{1}{2}$, the energy index of Σ_a equals four. In particular $\operatorname{ind}_E(\Sigma_a) > 3 \operatorname{ind}_S(\Sigma_a)$, and the inequality conjectured in* [10, Remark 7] *fails for every critical spherical catenoid in $\mathbb{H}^3$.*

In $\mathbb{S}^3_+$ the energy index of the critical spherical catenoid is three, by [10, Theorem 2, Proposition 11 and Theorem 18] and V. Lima's bound; in $\mathbb{H}^3$ it equals the Morse index.

Consequently, the upper bound in Theorem 1 cannot be obtained as in the spherical case: V. Lima's bound only gives $\operatorname{ind}(\Sigma_a) \le 5$. The fourth negative direction is explicit: it comes from the quadric cone with vertex e_0 through the two boundary circles (Lemma 18). For the critical catenoid in the Euclidean unit ball the corresponding field is null for the energy, and negative curvature makes it strictly negative (Remark 19). The proof is given in Section 9.

2. Geometry of the critical catenoids

We use the hyperboloid model

$$\mathbb{H}^3 = \{X \in \mathbb{R}^{3,1} : \langle X, X\rangle_L = -1,\ X^0 > 0\}, \qquad \langle X, Y\rangle_L = -X^0Y^0 + \sum_{j=1}^{3} X^jY^j.$$

The center of the balls under consideration is $e_0 = (1, 0, 0, 0)$. Fix $a > \frac{1}{2}$ and define

$$K = \sqrt{a^2 - \tfrac{1}{4}}, \qquad A(s)^2 = a\cosh(2s) + \tfrac{1}{2}, \quad B(s)^2 = a\cosh(2s) - \tfrac{1}{2}, \tag{2.1}$$

$$\varphi(s) = \int_0^s \frac{K}{A(\sigma)^2 B(\sigma)}\,\mathrm{d}\sigma. \tag{2.2}$$

The positive square roots are understood. The rotational parametrization is

$$X(s,\theta) = \big(A\cosh\varphi,\ A\sinh\varphi,\ B\cos\theta,\ B\sin\theta\big), \qquad (s,\theta) \in \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}). \tag{2.3}$$

A prime denotes differentiation with respect to s throughout the paper. Set $v = B'$. The identities

$$A^2 = 1 + B^2, \qquad A' = \frac{Bv}{A}, \qquad v = \frac{a\sinh(2s)}{B}, \tag{2.4}$$

$$v^2 = 1 + B^2 - \frac{K^2}{B^2}, \qquad v' = B + \frac{K^2}{B^3} > 0 \tag{2.5}$$

follow directly from (2.1). In particular, A and B are even, whereas v and φ are odd.

For completeness, introduce the Lorentz-orthonormal frame

$$E_0 = (\cosh\varphi, \sinh\varphi, 0, 0), \quad E_1 = (\sinh\varphi, \cosh\varphi, 0, 0), \quad E_r = (0, 0, \cos\theta, \sin\theta),$$

together with $E_\theta = (0, 0, -\sin\theta, \cos\theta)$. Then $X = AE_0 + BE_r$ and

$$X_s = A'E_0 + A\varphi' E_1 + vE_r, \qquad X_\theta = BE_\theta.$$

Using (2.2)–(2.5), we obtain

$$\langle X_s, X_s\rangle_L = \frac{v^2 + K^2/B^2}{A^2} = 1, \qquad \langle X_s, X_\theta\rangle_L = 0.$$

Consequently,

$$g = ds^2 + B^2 d\theta^2. \tag{2.6}$$

A unit normal is

$$\nu = \frac{K}{A}E_0 - \frac{v}{A}E_1 + \frac{K}{B}E_r. \tag{2.7}$$

Indeed, (2.2)–(2.5) imply that ν is orthogonal to X, X_s, X_θ and has Lorentzian norm one. A direct computation using (2.2)–(2.5) gives

$$\nu_s = -\frac{K}{B^2}X_s, \qquad \nu_\theta = \frac{K}{B^2}X_\theta.$$

Thus, with shape operator $-d\nu$, the principal curvatures are K/B^2 and $-K/B^2$. The surface is minimal and

$$|\mathrm{II}|^2 = \frac{2K^2}{B^4}. \tag{2.8}$$

The parametrization is embedded: the quotient $X^1/X^0 = \tanh\varphi(s)$ is strictly increasing in s, and, for each fixed s, the pair $(X^2, X^3) = B(s)(\cos\theta, \sin\theta)$ determines θ uniquely modulo 2π because $B(s) > 0$.

2.1. **Free boundary truncation.** Write

$$g_0(s) = A(s)\cosh\varphi(s) = X^0, \qquad g_1(s) = A(s)\sinh\varphi(s) = X^1.$$

For the distance ρ from e_0, one has $X^0 = \cosh\rho$ and

$$\nabla^{\mathbb{H}^3} X^0 = X^0 X - e_0, \qquad \langle \nabla^{\mathbb{H}^3} X^0, \nu\rangle_L = \nu^0.$$

It follows that the surface meets a sphere centered at e_0 orthogonally precisely when $\nu^0 = 0$. At a point $s = S > 0$, this condition reads

$$\tanh\varphi(S) = \frac{K}{v(S)}. \tag{2.9}$$

As functions of $S > 0$, the left-hand side increases strictly from zero, whereas the right-hand side decreases strictly from $+\infty$ to zero. Hence (2.9) has exactly one solution $S > 0$.

The critical catenoid is

$$\Sigma_a = X([-S, S] \times (\mathbb{R}/2\pi\mathbb{Z})).$$

It is symmetric with respect to the totally geodesic plane $\{X^1 = 0\}$, which contains the neck circle $s = 0$, and its supporting ball is centered at the point e_0 where this plane meets the axis $\{X^2 = X^3 = 0\}$. Set

$$B_0 = B(S), \qquad v_0 = v(S), \qquad r = \operatorname{arcosh} g_0(S), \qquad c = \coth r. \tag{2.10}$$

Since $g_0'(s) > 0$ for $s > 0$, every interior point of Σ_a lies strictly inside $\mathbb{B}(r)$. At the positive boundary the outward unit conormal is X_s. Orthogonality to the sphere gives

$$X_s(S, \theta) = \frac{\cosh r\, X(S, \theta) - e_0}{\sinh r}.$$

Taking components yields

$$g_0(S) = \cosh r, \qquad g_0'(S) = \sinh r, \qquad g_1'(S) = c\, g_1(S), \tag{2.11}$$

$$v_0 = cB_0, \qquad g_0'(S) - c\, g_0(S) = -\frac{1}{\sinh r}. \tag{2.12}$$

3. The quadratic form and separation of variables

We use the convention $\Delta = \operatorname{div}\nabla$. For the ambient Ricci tensor, $\mathrm{Ric}_{\mathbb{H}^3}(\nu, \nu) = -2$. If N is the outward unit normal to the supporting sphere, then $\langle \nabla^{\mathbb{H}^3}_V N, V\rangle_L = \coth r$ for every unit vector V tangent to that sphere. Along the free boundary, ν is such a tangent vector. In the sign convention of [17], the second fundamental form of the boundary is computed with respect to the outward normal, so that its value on ν is $-\coth r$. The general free boundary second variation formula [17, Section 2] (see also [10, Section 5.1]) therefore gives

$$Q[u] = \int_{\Sigma_a} \left(|\nabla u|^2 + (2 - |\mathrm{II}|^2)u^2\right) \mathrm{d}A - c\int_{\partial\Sigma_a} u^2\, \mathrm{d}\ell, \qquad u \in H^1(\Sigma_a). \tag{3.1}$$

Its associated self-adjoint operator and boundary condition are

$$J = -\Delta - |\mathrm{II}|^2 + 2, \qquad \partial_\eta u = cu,$$

where η is the outward unit conormal along $\partial\Sigma_a$. The index is the number of negative eigenvalues, counted with multiplicity; the nullity is the dimension of the kernel. The form domain is $H^1(\Sigma_a)$ and imposes no Robin condition on test functions.

For angular frequency $k \geq 0$, the radial operator is

$$J_k u = -\frac{1}{B}(Bu')' + \left(2 + \frac{k^2}{B^2} - \frac{2K^2}{B^4}\right) u \tag{3.2}$$

in $L^2((-S,S), B\,\mathrm{d}s)$, with boundary conditions

$$u'(S) = cu(S), \qquad -u'(-S) = cu(-S). \tag{3.3}$$

After removing the positive angular integration factor, the radial form is

$$\begin{aligned}\mathfrak{q}_k[u] = \int_{-S}^{S} \left(Bu'^2 + \left(2B + \frac{k^2}{B} - \frac{2K^2}{B^3} \right) u^2 \right) \mathrm{d}s \\ - v_0(u(S)^2 + u(-S)^2), \qquad u \in H^1(-S,S).\end{aligned} \tag{3.4}$$

Each $k \geq 1$ has the two real angular factors $\cos(k\theta)$ and $\sin(k\theta)$; the mode $k = 0$ has one. The coefficients are even in s, so each radial problem splits into even and odd sectors.

For each fixed $a > \frac{1}{2}$, one has $B(s) \geq \sqrt{a - \frac{1}{2}} > 0$ on the compact interval $[-S, S]$. All radial problems are therefore regular Sturm–Liouville problems with positive smooth weight. We use the min–max principle, simplicity of radial eigenvalues, and the oscillation theorem: the jth eigenfunction, with eigenvalues numbered starting at $j = 1$, has exactly $j - 1$ interior zeros. These standard facts are discussed in [18].

4. A global boundary inequality

For the flat connection D of $\mathbb{R}^{3,1}$, the Gauss formula reads

$$D_U(dX(V)) = dX(\nabla^{\Sigma}_U V) + \mathrm{II}(U,V)\nu + g(U,V)X.$$

Tracing and using minimality gives $\Delta_\Sigma X = 2X$. The coordinate functions consequently satisfy $\Delta_\Sigma X^j = 2X^j$, so

$$g_i'' + \frac{B'}{B} g_i' - 2g_i = 0, \qquad J_0 g_i = -|\mathrm{II}|^2 g_i, \qquad i = 0, 1. \tag{4.1}$$

Lemma 3. *For every $a > \frac{1}{2}$, the boundary radius in the rotational coordinates satisfies*

$$B_0^2 > 2K^2.$$

Proof. By (2.5),

$$v'' = v - \frac{3K^2 v}{B^4},$$

and substitution in (3.2) gives $J_0 v = 0$. Moreover, $v(0) = 0$ and $v > 0$ on $(0, S]$. Define

$$W = B(vg_1' - v'g_1).$$

The equations for v and g_1 imply

$$W(0) = 0, \qquad W' = B|\mathrm{II}|^2 v g_1 > 0 \quad \text{on } (0, S].$$

Using (2.11), we find

$$0 < W(S) = -B_0 g_1(S)\big(v'(S) - cv_0\big).$$

On the other hand, (2.5) and $c = v_0/B_0$ give

$$v'(S) - cv_0 = B_0 + \frac{K^2}{B_0^3} - \frac{v_0^2}{B_0} = \frac{2K^2 - B_0^2}{B_0^3}.$$

Since $B_0 g_1(S) > 0$, the asserted inequality follows. □

5. The modes zero and one

5.1. **A Dirichlet eigenfunction.**

Lemma 4. *The first Dirichlet eigenvalue of J_0 on $(-S, S)$ is 2. A corresponding positive eigenfunction is*

$$n(s) = \nu^0(s) = \frac{K\cosh\varphi(s) - v(s)\sinh\varphi(s)}{A(s)}.$$

Proof. The Gauss–Weingarten equations in $\mathbb{R}^{3,1}$ imply

$$\Delta_\Sigma \nu = -|\mathrm{II}|^2 \nu \tag{5.1}$$

for a minimal surface in $\mathbb{H}^3$. To see the signs explicitly, let $\mathcal{A}$ denote the shape operator. In a geodesic orthonormal frame on the surface, tracing the derivative of $D\nu = -dX \circ \mathcal{A}$ gives

$$\Delta_\Sigma \nu = -dX(\operatorname{div}\mathcal{A}) - |\mathrm{II}|^2\nu - (\operatorname{tr}\mathcal{A})X.$$

The Codazzi equation in constant curvature gives $\operatorname{div}\mathcal{A} = \nabla(\operatorname{tr}\mathcal{A})$, and both trace terms vanish by minimality. This proves (5.1). Taking its time component gives $J_0 n = 2n$.

The function n is even and $n(0) = K/A(0) > 0$. For $0 < s < S$, the uniqueness of the crossing in (2.9) gives $\tanh\varphi(s) < K/v(s)$, so $n(s) > 0$. At $s = \pm S$, the free boundary condition gives $n = 0$. Thus n is a Dirichlet eigenfunction with no interior zero. The oscillation theorem identifies its eigenvalue as the first one. □

Proposition 5. *The radial Robin problem for J_0 has exactly two negative eigenvalues and no zero eigenvalue. One negative eigenfunction is even and the other is odd.*

Proof. Integrating (4.1) by parts in (3.4), and using (2.11)–(2.12), gives

$$\mathfrak{q}_0[g_0] = -\int_{-S}^{S} B|\mathrm{II}|^2 g_0^2 \,\mathrm{d}s - 2B_0\coth r < 0, \tag{5.2}$$

$$\mathfrak{q}_0[g_1] = -\int_{-S}^{S} B|\mathrm{II}|^2 g_1^2 \,\mathrm{d}s < 0. \tag{5.3}$$

Since g_0 is even and g_1 is odd, their mixed term in the polarized quadratic form is zero. Hence their span is a two-dimensional negative subspace.

Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the radial Robin eigenvalues. The preceding subspace gives $\lambda_2 < 0$. The Dirichlet form domain $H^1_0(-S, S)$ has codimension two in $H^1(-S, S)$, since its defining conditions are the two endpoint traces. Therefore the span of the first three Robin eigenfunctions contains a nonzero $z \in H^1_0(-S, S)$. Lemma 4 and the min–max principle yield

$$2\int_{-S}^{S} Bz^2 \,\mathrm{d}s \le \mathfrak{q}_0[z] \le \lambda_3 \int_{-S}^{S} Bz^2 \,\mathrm{d}s.$$

Thus $\lambda_3 \ge 2$. This proves the exact count and excludes a kernel. The negative tests (5.2) and (5.3) supply one negative direction of each parity, so each parity contributes exactly one negative eigenvalue. □

5.2. **The first angular mode.**

Proposition 6. *The radial Robin problem for J_1 has one negative eigenvalue and a one-dimensional kernel generated by g_0'. The negative eigenfunction is even and the kernel function is odd.*

Proof. The identity

$$\left(\frac{B'}{B}\right)' = \frac{2K^2}{B^4} - \frac{1}{B^2}$$

shows, upon differentiating (4.1), that

$$J_1 g_0' = 0, \qquad J_1 g_1' = 0. \tag{5.4}$$

Set $f = g_0'$. The function f is odd, positive on $(0, S]$, and satisfies $f'(0) = 2g_0(0) > 0$. At $s = S$,

$$f'(S) = 2g_0(S) - c\, g_0'(S) = 2\cosh r - c\sinh r = \cosh r = c\, f(S).$$

Parity gives the other boundary condition in (3.3). Thus f is a Robin eigenfunction with eigenvalue zero and exactly one interior zero. By the oscillation theorem, zero is the second radial eigenvalue, and it is simple. There is exactly one negative eigenvalue.

A first eigenfunction can be chosen strictly positive. Reflection $s \mapsto -s$ preserves the problem, so simplicity makes that eigenfunction even. The kernel generator f is odd. □

6. Positivity of the higher angular modes

Define the conformal coordinate

$$t = t(s) = \int_0^s \frac{\mathrm{d}\sigma}{B(\sigma)}, \qquad T = t(S).$$

A dot denotes differentiation with respect to t; primes continue to denote differentiation with respect to s. We use the same letters for functions composed with $s(t)$. Equation (2.5) gives

$$\dot B = Bv, \qquad \dot v = \mathcal{D}(v) := \sqrt{(v^2-1)^2 + 4K^2}, \qquad v(0) = 0, \quad \mathcal{D}(0) = 2a. \tag{6.1}$$

Indeed, $\dot v = Bv' = B^2 + K^2/B^2 > 0$, and the square of this expression is $(v^2-1)^2 + 4K^2$.

In this coordinate the zero-eigenvalue equation is

$$\ddot u = (2v^2 + k^2 - 2)u, \tag{6.2}$$

and the full radial form is

$$\mathfrak{q}_k[u] = \int_{-T}^{T} (\dot u^2 + (2v^2 + k^2 - 2)u^2)\,\mathrm{d}t - v_0(u(T)^2 + u(-T)^2). \tag{6.3}$$

The norm in the spectral problem is

$$\int_{-S}^{S} Bu^2\,\mathrm{d}s = \int_{-T}^{T} B^2u^2\,\mathrm{d}t. \tag{6.4}$$

In particular, the conformal change preserves the form and the zero equation; the transformed eigenvalue problem has weight B^2.

6.1. **A boundary logarithmic derivative.**

Let u_2 solve

$$\ddot u_2 = (2v^2 + 2)u_2, \qquad u_2(0) = 1, \quad \dot u_2(0) = 0. \tag{6.5}$$

The coefficient is strictly positive. While u_2 is positive, $\ddot u_2 > 0$, so u_2 increases from 1 and cannot have a first zero. Hence $u_2 > 0$ on $[0, T]$ and $\dot u_2 > 0$ on $(0, T]$. Define

$$p_2 = \frac{\dot u_2}{u_2}, \qquad \delta_2 = p_2 - v.$$

Lemma 7. *For every $a > \frac{1}{2}$, one has $p_2(T) > v_0$.*

Proof. The Riccati equation for p_2 and (6.1) give

$$\dot\delta_2 + (p_2 + v)\delta_2 = F_2(v), \qquad \delta_2(0) = 0, \qquad F_2(v) = v^2 + 2 - \mathcal{D}(v). \tag{6.6}$$

Since $(v^2+2)^2 - \mathcal{D}(v)^2 = 6v^2 + 4 - 4a^2$, we have the exact identity

$$F_2(v) = \frac{6v^2 + 4 - 4a^2}{v^2 + 2 + \mathcal{D}(v)}. \tag{6.7}$$

If $a \le 1$, then $F_2(v(t)) > 0$ for every $t > 0$. An integrating factor in (6.6) yields $\delta_2(T) > 0$.

Suppose now that $a > 1$. By (6.7), $F_2(v)$ is negative for $v^2 < 2(a^2-1)/3$ and positive for $v^2 > 2(a^2-1)/3$. Since v is strictly increasing, the function

$$M(t)\delta_2(t) = \int_0^t M(\tau)F_2(v(\tau))\,\mathrm{d}\tau, \qquad M(t) = \exp\left(\int_0^t (p_2+v)\,\mathrm{d}\tau\right) > 0,$$

first decreases and then increases, if the sign change is reached. It follows that

$$\delta_2(T) \le 0 \quad \Longrightarrow \quad \delta_2(t) < 0 \quad (0 < t < T). \tag{6.8}$$

If the sign change is not reached before T, the same implication follows from strict decrease throughout the interval.

For a comparison solution, set $w = g_1'$. From the definitions,

$$w = A'\sinh\varphi + A\varphi'\cosh\varphi > 0 \qquad (s \ge 0).$$

The function w is even, and $J_1 w = 0$ by (5.4). Consequently, for

$$p_1 = \frac{\dot w}{w}, \qquad \delta_1 = p_1 - v,$$

we have

$$\dot p_1 = 2v^2 - 1 - p_1^2, \qquad p_1(0) = 0,$$

and

$$\dot\delta_1 + (p_1+v)\delta_1 = v^2 - 1 - \mathcal{D}(v) < 0, \qquad \delta_1(0) = 0.$$

The strict sign holds because $K > 0$, so $\mathcal{D}(v) > |v^2-1|$. Thus $p_1 < v$ for $t > 0$.

Assume, for contradiction, that $\delta_2(T) \le 0$. By (6.8), $p_1 + p_2 < 2v$ on $(0,T)$. The difference $\omega = p_2 - p_1$ satisfies

$$\dot\omega + (p_1+p_2)\omega = 3, \qquad \omega(0) = 0.$$

Therefore

$$\begin{aligned}\omega(T) &= 3\int_0^T \exp\left(-\int_\tau^T (p_1+p_2)\,\mathrm{d}t\right)\mathrm{d}\tau \\ &> \frac{3}{B_0^2}\int_0^T B(t)^2\,\mathrm{d}t = \frac{3}{B_0^2}\int_0^S B(s)\,\mathrm{d}s. \end{aligned} \tag{6.9}$$

Here we used $\dot B/B = v$; the strict inequality holds on every integration interval with $\tau < T$.

At the boundary, (4.1) and $g_1'(S) = (v_0/B_0)g_1(S)$ give

$$p_1(T) = \frac{B_0 g_1''(S)}{g_1'(S)} = \frac{2B_0^2}{v_0} - v_0.$$

Since $p_2(T) \le v_0$, equation (2.5) yields

$$\omega(T) \le v_0 - p_1(T) = \frac{2(B_0^2 - K^2)}{B_0^2 v_0}. \tag{6.10}$$

We now prove that (6.9) and (6.10) are incompatible. For $s > 0$, define

$$\mathcal{I}(s) = 3\int_0^s B(\sigma)\,\mathrm{d}\sigma - \frac{2(B(s)^2 - K^2)}{v(s)}.$$

Let $x = B^2$ and $\kappa = K^2$. Direct differentiation gives

$$\begin{aligned}\mathcal{I}' &= -B + \frac{2(B^2-\kappa)(B+\kappa/B^3)}{v^2} \\ &= B\left(-1 + \frac{2(x-\kappa)(x^2+\kappa)}{x(x^2+x-\kappa)}\right) \\ &= B\,\frac{(x-2\kappa)(x^2-x+\kappa)}{x(x^2+x-\kappa)}. \end{aligned} \tag{6.11}$$

For the last step, the numerator expands as

$$\begin{aligned} &-x(x^2+x-\kappa)+2(x-\kappa)(x^2+\kappa)\\ &\quad= x^3-(1+2\kappa)x^2+3\kappa x-2\kappa^2\\ &\quad= (x-2\kappa)(x^2-x+\kappa). \end{aligned}$$

The denominator in (6.11) is positive for $s>0$, because $x^2+x-\kappa = xv^2>0$. Since $a>1$, we have $\kappa>\frac{3}{4}$, and

$$x^2-x+\kappa = (x-\tfrac{1}{2})^2+\kappa-\tfrac{1}{4}>0.$$

Set

$$s_* = \tfrac{1}{2}\operatorname{arcosh}(2a), \qquad B(s_*)=\sqrt{2}K, \qquad v(s_*)=\sqrt{2}a. \tag{6.12}$$

Lemma 3 implies $S>s_*$. Equation (6.11) shows that $\mathcal{I}$ is strictly increasing on $(s_*,S]$.

The identity

$$\left(\frac{B'}{B}\right)^2 = 1+\frac{1}{x}-\frac{K^2}{x^2} = \frac{a^2}{K^2}-\frac{(x-2K^2)^2}{4K^2x^2} \le \frac{a^2}{K^2} \tag{6.13}$$

follows by expanding the square and using $a^2=K^2+\frac{1}{4}$. As $B'\ge 0$ for $s\ge 0$, integration of $B'\le (a/K)B$ gives

$$\int_0^{s_*} B\,\mathrm{d}s \ge \frac{K}{a}\left(\sqrt{2}K-\sqrt{a-\tfrac{1}{2}}\right).$$

Using (6.12), we conclude that

$$\mathcal{I}(s_*) \ge \frac{K}{a}\left(2\sqrt{2}K-3\sqrt{a-\tfrac{1}{2}}\right)>0. \tag{6.14}$$

The final strict inequality follows from

$$8K^2-9(a-\tfrac{1}{2}) = (a-\tfrac{1}{2})(8a-5)>0.$$

Thus $\mathcal{I}(S)>0$. In contrast, (6.9)–(6.10) imply

$$3\int_0^S B\,\mathrm{d}s < \frac{2(B_0^2-K^2)}{v_0},$$

which says $\mathcal{I}(S)<0$. This contradiction proves $\delta_2(T)>0$ also when $a>1$. □

6.2. **A positive quadratic form.**

Proposition 8. *Every radial Robin eigenvalue in angular mode $k\ge 2$ is strictly positive.*

Proof. Extend u_2 evenly to $[-T,T]$. The coefficient in (6.5) is even, so the extension is smooth, positive, and solves the same equation. Its logarithmic derivative p_2 is odd. For any $z\in H^1(-T,T)$, writing $z=u_2h$ and integrating

$$\dot z^2+(2v^2+2)z^2 = u_2^2\dot h^2+\frac{\mathrm{d}}{\mathrm{d}t}(p_2z^2)$$

gives

$$\begin{aligned} \mathfrak{q}_2[z] = &\int_{-T}^{T} u_2^2\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{z}{u_2}\right)^2\mathrm{d}t\\ &+\big(p_2(T)-v_0\big)\big(z(T)^2+z(-T)^2\big). \end{aligned} \tag{6.15}$$

The identity first follows for smooth functions and extends to H^1 by density and continuity of endpoint traces. Lemma 7 makes the boundary coefficient strictly positive. If the right-hand side vanishes, then z/u_2 is constant and $z(T)=z(-T)=0$, which forces $z=0$. Therefore $\mathfrak{q}_2[z]>0$ for every nonzero $z\in H^1(-T,T)$.

The minimum of the Rayleigh quotient with the positive weight $B^2\,\mathrm{d}t$ in (6.4) is attained for this regular problem. Hence its first eigenvalue is strictly positive. Finally,

$$\mathfrak{q}_k[z] = \mathfrak{q}_2[z] + (k^2-4)\int_{-T}^{T} z^2\,\mathrm{d}t \qquad (k \geq 2)$$

proves the assertion for all higher modes, on the full form domain and therefore in both parity sectors. □

7. Proof of the main theorem

Proof of Theorem 1. Propositions 5, 6, and 8 give the following contributions. Angular multiplicities are included in the table.

Sector	Index contribution	Nullity contribution
$k=0$, even	1	0
$k=0$, odd	1	0
$k=1$, even	2	0
$k=1$, odd	0	2
$k\geq 2$, both parities	0	0

The angular Fourier decomposition is orthogonal for both the L^2 inner product and the quadratic form. Its partial sums converge in $H^1(\Sigma_a)$, and hence their boundary traces converge in $L^2(\partial\Sigma_a)$. Thus the decomposition also holds on the full form domain. The span of the negative eigenfunctions in the table has dimension four, and its L^2-orthogonal complement is nonnegative for Q. A function in the Robin kernel has each Fourier coefficient in the corresponding radial kernel. The mode counts therefore give

$$\operatorname{ind}(\Sigma_a) = 4, \qquad \ker J = \operatorname{span}\{g_0'(s)\cos\theta,\ g_0'(s)\sin\theta\}.$$

In particular, $\operatorname{nul}(\Sigma_a) = 2$.

To identify the kernel geometrically, consider the spatial rotation fields

$$Z_2(X) = (0, -X^2, X^1, 0), \qquad Z_3(X) = (0, -X^3, 0, X^1).$$

They preserve the center e_0 and the geodesic ball. Equation (2.7) gives

$$\langle Z_2, \nu\rangle_L = g_0'\cos\theta, \qquad \langle Z_3, \nu\rangle_L = g_0'\sin\theta.$$

Indeed, $\langle Z_2,\nu\rangle_L = (-B\nu^1 + (K/B)g_1)\cos\theta$ and $\langle Z_3,\nu\rangle_L = (-B\nu^1+(K/B)g_1)\sin\theta$, while

$$-B\nu^1 + \frac{K}{B}g_1 = \frac{Bv}{A}\cosh\varphi + \frac{K}{AB}\sinh\varphi = g_0'.$$

These rotations tilt the axis of the catenoid. Rotation about the axis itself has zero normal component. The two displayed fields span the entire Robin kernel, as asserted. □

Remark 9. For each fixed $a > \frac{1}{2}$, the theorem gives nondegeneracy modulo the rotations preserving the supporting ball. All spectral comparisons above are made for that fixed parameter; no uniform spectral gap as the parameter approaches an endpoint is required. In the terminology of [1], Σ_a is equivariantly nondegenerate; see Corollary 12.

8. Consequences

Corollary 10. *The supporting radius $r(a)$ depends smoothly on a and satisfies*

$$\frac{\mathrm{d}r}{\mathrm{d}a} > 0 \qquad \textit{for every } a > \tfrac{1}{2}.$$

In particular, distinct members of the family $\{\Sigma_a\}_{a>1/2}$ are free boundary in distinct geodesic balls centered at e_0.

Proof. For $s > 0$, the function $\tanh\varphi - K/v$ has s-derivative $\varphi' \operatorname{sech}^2\varphi + Kv'/v^2 > 0$. By the implicit function theorem, the solution S of (2.9) depends smoothly on a, and so does $r = \operatorname{arcosh} g_0(S)$, since $g_0(S) > 1$.

All quantities of Section 2 depend smoothly on (s, a). Let $Y = \partial_a X$, taken at fixed (s, θ), and set $\psi = \langle Y, \nu\rangle_L$. This is the parametric Jacobi field of [14, Definition 8.1]; its evenness, the value $\psi(0) = 1/(2K)$ and the boundary identity (8.1) below are established in [14, Lemmas 8.2 and 8.3], and we include the short derivations for completeness. Each $X(\cdot, \cdot\,; a)$ is a minimal immersion of $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$, so the normal component of the variation field is a Jacobi field. Since ψ does not depend on θ, this gives $J_0\psi = 0$. The Lorentz reflection $R = \operatorname{diag}(1, -1, 1, 1)$ satisfies $X(-s, \theta) = RX(s, \theta)$ and $\nu(-s, \theta) = R\nu(s, \theta)$ for every a; hence ψ is even.

At $s = 0$ one has $\varphi = \partial_a\varphi = 0$ and $v = 0$. Since $\partial_a A^2 = \partial_a B^2 = \cosh(2s)$, the vectors Y and ν at $s = 0$ are

$$\Big(\frac{1}{2A}, 0, \frac{\cos\theta}{2B}, \frac{\sin\theta}{2B}\Big) \quad \text{and} \quad \Big(\frac{K}{A}, 0, \frac{K}{B}\cos\theta, \frac{K}{B}\sin\theta\Big),$$

with $A = A(0)$ and $B = B(0)$. Using $A(0)^2 - B(0)^2 = 1$ and $A(0)^2 B(0)^2 = K^2$, we obtain

$$\psi(0) = \frac{K}{2}\Big(\frac{1}{B(0)^2} - \frac{1}{A(0)^2}\Big) = \frac{1}{2K} \neq 0.$$

Now let $s = S$. The boundary relation of Section 2 gives $e_0 = \cosh r\, X - \sinh r\, X_s$; in particular $\langle X_s, e_0\rangle_L = -\sinh r$. Differentiating $\langle X, X\rangle_L = -1$, $\langle \nu, X\rangle_L = 0$ and $\langle \nu, X_s\rangle_L = 0$ in a, and using $\nu_s = -(K/B^2)X_s$, we find

$$\langle Y, X\rangle_L = 0, \qquad \langle \partial_a\nu, X\rangle_L = -\psi, \qquad \langle \partial_a\nu, X_s\rangle_L = -\psi' - \frac{K}{B^2}\langle Y, X_s\rangle_L.$$

Since $\partial_a g_0 = -\langle Y, e_0\rangle_L = \sinh r\, \langle Y, X_s\rangle_L$ at $s = S$ and $g_0'(S) = \sinh r$, differentiating $g_0(S(a)) = \cosh r(a)$ and dividing by $\sinh r$ gives

$$\frac{\mathrm{d}S}{\mathrm{d}a} + \langle Y, X_s\rangle_L = \frac{\mathrm{d}r}{\mathrm{d}a}.$$

Since $\langle \nu_s, e_0\rangle_L = (K/B_0^2)\sinh r$ at $s = S$, differentiating the free boundary condition $\langle \nu, e_0\rangle_L|_{s=S(a)} = -\nu^0(S(a)) = 0$, and using the decomposition of e_0 together with the identities above, gives

$$\frac{K}{B_0^2}\sinh r\,\frac{\mathrm{d}S}{\mathrm{d}a} - \cosh r\,\psi(S) + \sinh r\Big(\psi'(S) + \frac{K}{B_0^2}\langle Y, X_s\rangle_L\Big) = 0.$$

Dividing by $\sinh r$ and combining the last two identities, we obtain

$$\psi'(S) - c\,\psi(S) = -\frac{K}{B_0^2}\,\frac{\mathrm{d}r}{\mathrm{d}a}. \tag{8.1}$$

Suppose that $\mathrm{d}r/\mathrm{d}a = 0$ for some a. Then (8.1) and the evenness of ψ show that ψ satisfies both conditions in (3.3). Since $J_0\psi = 0$ and $\psi(0) \neq 0$, zero would be an eigenvalue of the radial Robin problem for J_0, contrary to Proposition 5. Hence $\mathrm{d}r/\mathrm{d}a$ never vanishes and has constant sign. By Lemma 3,

$$\cosh r = A(S)\cosh\varphi(S) \geq \sqrt{1 + B_0^2} > \sqrt{1 + 2K^2},$$

and the right-hand side tends to infinity as $a \to \infty$. Thus r is unbounded as $a \to \infty$, which excludes the negative sign. □

Lemma 11. *$S(a) \to 0$ and $r(a) \to 0$ as $a \to \frac{1}{2}^+$.*

Proof. Put $\epsilon = a - \frac{1}{2} \in (0, \frac{1}{2})$. Then $K^2 = \epsilon(1+\epsilon) \in (\epsilon, 2\epsilon)$ and $\epsilon + \sinh^2 s \le B(s)^2 = \epsilon + 2a\sinh^2 s \le \epsilon + 2\sinh^2 s$. Fix $s_1 \in (0,1]$ and suppose that $S \ge s_1$. As v is increasing and $v(s_1) \ge c_1 := \sinh(2s_1)/(2\sqrt{1/2 + 2\sinh^2 s_1}) > 0$, (2.9) gives

$$\tanh\varphi(s_1) \le \tanh\varphi(S) = \frac{K}{v(S)} \le \frac{\sqrt{2\epsilon}}{c_1}.$$

On $[0,1]$ one has $A^2 \le c_2 := \frac{3}{2} + 2\sinh^2 1$ and, since $\sinh\sigma \le \sigma \sinh 1$ there, $B(\sigma)^2 \le \epsilon + c_3\sigma^2$ with $c_3 = 2\sinh^2 1$. Hence

$$\varphi(s_1) \ge \frac{\sqrt{\epsilon}}{c_2}\int_0^{s_1} \frac{\mathrm{d}\sigma}{\sqrt{\epsilon + c_3\sigma^2}} = \frac{\sqrt{\epsilon}}{c_2\sqrt{c_3}}\,\operatorname{arsinh}\Big(s_1\sqrt{c_3/\epsilon}\Big),$$

while $\varphi(s_1) \le K\int_0^1 (\epsilon+\sigma^2)^{-1/2}\,\mathrm{d}\sigma \to 0$. Since $\tanh x \ge x/2$ on $[0,1]$, the quotient $\tanh\varphi(s_1)/\sqrt{\epsilon}$ tends to $+\infty$ as $\epsilon \to 0$, which contradicts the previous display for ϵ small. Hence $S(a) < s_1$ for a close to $\frac{1}{2}$, that is, $S(a) \to 0$. Finally, $\varphi(S) \le K\int_0^S B^{-1}\,\mathrm{d}\sigma \le \sqrt{2}\,S \to 0$ and $A(S)^2 = 1 + B(S)^2 \to 1$, so $\cosh r(a) = A(S)\cosh\varphi(S) \to 1$. □

Combined with the limit $r(a) \to 0$ as $a \to \frac{1}{2}^+$ (Lemma 11; see also [15]), Corollary 10 shows that $a \mapsto r(a)$ is an increasing diffeomorphism of $(\frac{1}{2}, \infty)$ onto $(0,\infty)$. Thus every geodesic ball of $\mathbb{H}^3$ is the supporting ball of an isometric image of exactly one Σ_a. This was stated without proof by Li and Xiong [6, Section 2]. Corollary 10 answers the question of the strict monotonicity of $r(a)$; see [14, Remark 1.3].

Corollary 12 (Equivariant nondegeneracy and CMC deformations)**.** *For every $a > \frac{1}{2}$, the critical spherical catenoid $\Sigma_a \subset \mathbb{B}(r(a)) \subset \mathbb{H}^3$ is equivariantly nondegenerate in the sense of* [1, Definition 2.4]*. Consequently, there exist $\varepsilon = \varepsilon(a) > 0$ and a smooth family $\{\Sigma_{a,h}\}_{|h|<\varepsilon}$ of embedded free boundary annuli in the fixed ball $\mathbb{B}(r(a))$ such that $\Sigma_{a,0} = \Sigma_a$ and $\Sigma_{a,h}$ has constant mean curvature h; here the mean curvature is the trace of the second fundamental form with respect to the unit normal extending ν. This family is locally unique up to congruence: a free boundary surface in $\mathbb{B}(r(a))$ sufficiently close to Σ_a in $C^{2,\alpha}$, with constant mean curvature h' with respect to the unit normal close to ν and with $|h'|$ sufficiently small, is congruent to $\Sigma_{a,h'}$ in the sense of* [1, Section 2.4].

Proof. Fix $a > \frac{1}{2}$, let $M = \mathbb{B}(r(a))$ carry the hyperbolic metric g, and orient Σ_a by ν. We verify the hypotheses of [1, Theorem 5.1].

The surface. By Section 2, $\Sigma_a \cap \partial M = \partial\Sigma_a$; this is condition (a) of [1, Definition 2.1]. Since ν is a global unit normal, the normal bundle of Σ_a is orientable; this is condition (b). Hence Σ_a is admissible. At $s = \pm S$ the outward unit normal of ∂M is $\pm X_s$, which is tangent to Σ_a; thus Σ_a is normal in the sense of the same definition. Since $H_1(M;\mathbb{Z}_2) = 0$ and Σ_a is a compact, connected, two-sided surface with $\Sigma_a \cap \partial M = \partial\Sigma_a$, the set $M \setminus \Sigma_a$ has exactly two connected components, and both have compact closure. Each geodesic segment in $\{X^1 = 0\}$ from e_0 to ∂M meets Σ_a exactly once, transversally, on the neck circle $s = 0$, and there ν points away from e_0. Hence e_0 and the circle $\partial M \cap \{X^1 = 0\}$ lie in different components, and, Σ_a being connected, ν points everywhere into the component Ω_1 that contains $\partial M \cap \{X^1 = 0\}$. With this choice of Ω_1, condition (c) of [1, Definition 2.1] holds, so Σ_a bounds a finite volume. Moreover, the unit normal used in [1], which points into Ω_1, is ν; hence the mean curvature in [1], the trace of the second fundamental form with respect to this normal, is the one in the statement.

Killing fields. The rotations $Z_1(X) = (0, 0, -X^3, X^2)$, Z_2 and Z_3 of Section 7 fix e_0; hence they are Killing fields of g tangent to ∂M, as required in [1, Section 2.4]. Conversely, since $\mathbb{H}^3$ is simply connected and has constant curvature, a Killing field defined on a connected open set is the restriction of a global one. If it is tangent to ∂M on a nonempty open subset of ∂M, then by analyticity it is tangent to the whole sphere ∂M, its flow preserves ∂M, and

therefore it vanishes at e_0; that is, it lies in $\operatorname{span}\{Z_1, Z_2, Z_3\}$. As $\langle Z_1, \nu\rangle_L = 0$, the space of Killing–Jacobi fields along Σ_a is

$$\operatorname{span}\{\langle Z_2, \nu\rangle_L, \langle Z_3, \nu\rangle_L\} = \operatorname{span}\{g_0' \cos\theta,\ g_0' \sin\theta\},$$

which is two-dimensional because $g_0' > 0$ on $(0, S]$.

Equivariant nondegeneracy. In [1] the second fundamental form of ∂M is taken with respect to its outer unit normal; for the geodesic sphere of radius $r(a)$ this gives $\mathrm{II}^{\partial M}(\nu, \nu) = -\coth r(a)$ along $\partial\Sigma_a$. Hence the linearized free boundary condition [1, (2.3)] is the Robin condition $\partial_\eta u = cu$ of Section 3, and the space of Jacobi fields in $C^{2,\alpha}_{\partial}(\Sigma_a)$ is the Robin kernel of Theorem 1. By that theorem, it coincides with the space of Killing–Jacobi fields. Thus Σ_a is equivariantly nondegenerate.

Deformation. Apply [1, Theorem 5.1] with $\Lambda = \mathbb{R}$, the constant family $g_\lambda \equiv g$, so that all g_λ define the same normal bundle of ∂M, an arbitrary $\lambda_0 \in \Lambda$, $h_0 = 0$, and the constant frame of Killing–Jacobi field generators $K_1^\lambda = Z_2$, $K_2^\lambda = Z_3$. The theorem gives a smooth family $\Sigma_{(h,\lambda)}$, for (h, λ) near $(0, \lambda_0)$, of free boundary hypersurfaces of M diffeomorphic to Σ_a, with constant mean curvature h and $\Sigma_{(0,\lambda_0)} = \Sigma_a$, which is unique modulo congruence. Setting $\Sigma_{a,h} = \Sigma_{(h,\lambda_0)}$ for $|h| < \varepsilon$ proves the existence statement. Since g_λ does not depend on λ, the uniqueness clause of [1, Theorem 5.1] with $\lambda' = \lambda_0$ is the asserted local uniqueness. □

Remark 13. For the critical catenoid in the Euclidean unit ball, equivariant nondegeneracy and a two-parameter family of free boundary CMC annuli in the unit balls of the space forms of curvature λ close to 0 were obtained by Bettiol, Piccione and Santoro [1, Lemma 8.1 and Proposition 8.3]; see also [1, Remark 8.2]. After rescaling, the members with $\lambda < 0$ of that family lie in geodesic balls of $\mathbb{H}^3$ of small radius. Corollary 12 applies to every $a > \frac{1}{2}$ and hence, by Corollary 10 and the discussion following it, to the critical spherical catenoid in every geodesic ball of $\mathbb{H}^3$. Embedded, non-rotational free boundary CMC annuli in geodesic balls of $\mathbb{H}^3$, with mean curvature $H > 1$ in their normalization, were constructed by Cerezo, Fernández and Mira [4].

Remark 14. The restriction to angular period 2π in Theorem 1 cannot be dropped. The radial form (3.4) is defined for every real $k \geq 0$, and

$$\mathfrak{q}_k[u] - \mathfrak{q}_\ell[u] = (k^2 - \ell^2) \int_{-S}^{S} \frac{u^2}{B}\, \mathrm{d}s.$$

Since $B(0) \leq B \leq B_0$ on $[-S, S]$, the min–max principle gives, for the radial Robin eigenvalues $\lambda_j(k)$,

$$\frac{k^2 - \ell^2}{B_0^2} \leq \lambda_j(k) - \lambda_j(\ell) \leq \frac{k^2 - \ell^2}{B(0)^2} \qquad (0 \leq \ell < k).$$

Hence each $\lambda_j(k)$ is a continuous, strictly increasing function of $k \geq 0$. Propositions 6 and 8 give $\lambda_1(1) < 0 < \lambda_1(2)$, so there is a unique $k_*(a) \in (1, 2)$ with $\lambda_1(k_*(a)) = 0$. For $0 < k < 1$, Propositions 5 and 6 give $\lambda_1(k) < \lambda_2(k) < \lambda_2(1) = 0 < \lambda_3(0) < \lambda_3(k)$; for $1 < k < 2$, one has $\lambda_2(k) > \lambda_2(1) = 0$. On the m-fold cover of Σ_a, of angular period $2\pi m$, the angular frequencies are j/m with integer $j \geq 0$. Each frequency $j/m \in (0, 1)$ therefore contributes four negative directions, so for $m \geq 2$ the index of the cover is at least $4m$. Moreover, the nullity of the cover exceeds two exactly when $mk_*(a)$ is an integer, which again requires $m \geq 2$. This agrees with the construction of Cerezo [3], in which non-rotational free boundary minimal annuli in geodesic balls of $\mathbb{H}^3$ bifurcate from m-fold covers of critical spherical catenoids (called free boundary hyperbolic catenoids in [3]). The bifurcating annuli are invariant under n reflections in planes through the axis, where m and n are coprime and n/m lies in an interval contained in $(\sqrt{2}, \sqrt{3}) \subset (1, 2)$.

Remark 15 (Scope of the argument). Lemmas 3 and 7 rest on identities specific to the rotational family of Section 2, and the separation of variables uses its rotational symmetry. Three ingredients nevertheless have a wider scope.

(i) The identity (5.1) uses neither the explicit parametrization nor the rotational symmetry: for every two-sided minimal hypersurface $\Sigma^n \subset \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1,1}$, the Gauss–Weingarten and Codazzi equations give $\Delta_\Sigma \nu = -|\mathrm{II}|^2 \nu$, hence $J\nu^0 = n\,\nu^0$ for $J = -\Delta_\Sigma - |\mathrm{II}|^2 + n$. If Σ is free boundary in a geodesic ball centered at e_0, then $\nu^0 = 0$ on $\partial\Sigma$. What is specific to the present family is the sign property of Lemma 4: ν^0 is positive in the interior, hence it is the first Dirichlet eigenfunction, with eigenvalue 2; this is the input of Proposition 5.

(ii) The last step of Proposition 8 is an instance of the ground-state representation for regular Sturm–Liouville problems. Write the Robin conditions as $\partial_\eta z = \sigma z$ at the endpoints. If a positive solution u of the zero-energy equation satisfies $\partial_\eta u/u > \sigma$ at both endpoints, the substitution $z = uh$ writes the quadratic form as a nonnegative interior square plus boundary terms whose coefficients are positive multiples of $\partial_\eta u/u - \sigma$; the form is therefore positive definite. By symmetry the two conditions reduce here to $p_2(T) > v_0$, which is Lemma 7.

(iii) The proof of Corollary 10 reflects a general transversality principle. Let X_a be a smooth family of free boundary minimal hypersurfaces in the geodesic balls of radius $r(a)$ centered at e_0, parametrized over a fixed manifold with boundary, and let $\psi = \langle \partial_a X_a, \nu \rangle_L$. Then ψ is a Jacobi field, and differentiating the two free boundary conditions gives

$$\partial_\eta \psi - \coth r\, \psi = -\mathrm{II}(\eta,\eta)\, \frac{\mathrm{d}r}{\mathrm{d}a} \qquad \text{on } \partial\Sigma,$$

where η is the outward conormal; here one uses that η is a principal direction along $\partial\Sigma$, the supporting sphere being umbilic. This is the Robin defect identity of [13, Theorem 3.1], which holds in all space forms and dimensions, with variants for warped products and capillary hypersurfaces [13, Remarks 3.2 and 3.3]; there the second fundamental form is $-\mathrm{II}$. For Σ_a one has $\mathrm{II}(\eta,\eta) = K/B_0^2$, and this is (8.1). Hence, if $\mathrm{d}r/\mathrm{d}a$ vanished, ψ would lie in the Robin kernel. For Σ_a, ψ is nonzero since $\psi(0) = 1/(2K)$, and Proposition 5 excludes a radial Robin kernel, so $\mathrm{d}r/\mathrm{d}a \neq 0$; the sign is then fixed by the unboundedness of r.

9. The energy index

Fix $a > \frac{1}{2}$ and write $\Sigma = \Sigma_a$, $r = r(a)$ and $c = \coth r$, as in (2.10). The flat connection D of $\mathbb{R}^{3,1}$ acts componentwise on $\mathbb{R}^{3,1}$-valued functions along Σ; for such functions U, V set

$$\langle \nabla U, \nabla V \rangle_L = \sum_{i=1}^{2} \langle D_{e_i} U, D_{e_i} V \rangle_L$$

for a local orthonormal frame $\{e_1, e_2\}$ of $T\Sigma$. We write $\hat{X} = (X^1, X^2, X^3)$, so that $X = (X^0, \hat{X})$, and $dX(W)$ for the image of a tangent vector field W of Σ, as in the Gauss formula of Section 4. The shape operator $\mathcal{A}$ satisfies $D\nu = -dX \circ \mathcal{A}$ and $\mathrm{II}(U,V) = g(\mathcal{A}U, V)$. As in Section 3, η is the outward unit conormal along $\partial\Sigma$. By Section 2 and the reflection symmetry,

$$\eta = \frac{\cosh r\, X - e_0}{\sinh r} = cX - \frac{e_0}{\sinh r} \qquad \text{along } \partial\Sigma. \tag{9.1}$$

9.1. **Energy index and spectral index.** Let

$$\mathcal{V} = \big\{ V \in C^\infty(\Sigma; \mathbb{R}^{3,1}) : \langle V, X \rangle_L = 0 \text{ on } \Sigma, \quad \langle V, e_0 \rangle_L = 0 \text{ on } \partial\Sigma \big\},$$

and

$$S_E(U,V) = \int_\Sigma \big(\langle \nabla U, \nabla V \rangle_L + 2\langle U, V \rangle_L \big)\, \mathrm{d}A - c \int_{\partial\Sigma} \langle U, V \rangle_L\, \mathrm{d}\ell. \tag{9.2}$$

The energy index $\mathrm{ind}_E(\Sigma)$ is the maximal dimension of a subspace of $\mathcal{V}$ on which S_E is negative definite. Since S_E depends only on the restriction to Σ, and every $V \in \mathcal{V}$ extends to a smooth field on $\mathbb{B}(r)$ tangent to $\mathbb{H}^3$ whose e_0-component vanishes on $\partial\mathbb{B}(r)$, this is the energy index of [10, Definition 2]. To construct the extension, let V_1 be the orthogonal projection onto $T\mathbb{H}^3$ of a smooth extension of V to $\mathbb{B}(r)$; then $V_1 = V$ on Σ. Let ρ be the

distance from e_0, and let χ be a cutoff function that vanishes near e_0 and equals one near $\partial\mathbb{B}(r)$. Since Σ meets $\partial\mathbb{B}(r)$ transversally along $\partial\Sigma$, where $\langle V_1, e_0\rangle_L = 0$, there is a smooth function h on $\mathbb{B}(r)$ that vanishes on Σ and equals $\langle V_1, e_0\rangle_L$ on $\partial\mathbb{B}(r)$. As $\langle \partial_\rho, e_0\rangle_L = -\sinh\rho$, the field $V_1 + \chi h\,(\sinh\rho)^{-1}\partial_\rho$ is the required extension.

The spectral index $\mathrm{ind}_S(\Sigma)$ is the number, counted with multiplicity, of eigenvalues $\sigma < c$ of the Steklov-type problem $\Delta_\Sigma u = 2u$ in Σ, $\partial_\eta u = \sigma u$ on $\partial\Sigma$ [10, Definition 3]. By the Dirichlet principle and the positivity of $\mathcal{R}$ on $H^1_0(\Sigma)$, it is also the index on $H^1(\Sigma)$ of the form

$$\mathcal{R}(u,w) = \int_\Sigma \left(\langle\nabla u, \nabla w\rangle + 2uw\right) \mathrm{d}A - c\int_{\partial\Sigma} uw\,\mathrm{d}\ell. \tag{9.3}$$

For Σ_a it equals one [10, Theorem 18]. This also follows from Section 3. After separation of variables, the radial forms of $\mathcal{R}$ are those in (3.4) without the term $-2K^2u^2/B^3$, and the Sturm–Liouville facts recalled there apply. For $k = 0$, the function g_1 solves the radial equation by (4.1) and satisfies (3.3) by (2.11), since it is odd. It vanishes only at $s = 0$, because $g_1 = A\sinh\varphi$ and φ is odd and strictly increasing. Hence g_1 is the second eigenfunction, with eigenvalue zero, and the mode $k = 0$ contributes exactly one negative eigenvalue. For $k = 1$, the function B solves the radial equation, since $X^2 = B\cos\theta$ and $\Delta_\Sigma X^2 = 2X^2$, and it satisfies (3.3), since $v_0 = cB_0$ by (2.12) and v is odd. As $B > 0$, the lowest eigenvalue for $k = 1$ is zero. For $k \geq 2$ the radial forms exceed the one for $k = 1$ and are therefore positive. Hence $\mathrm{ind}_S(\Sigma_a) = 1$.

Remark 16. The form (9.2) is formula (9) in [10] with $n = 3$ and $k = 2$. There $|\nabla V|^2 = -|\nabla V^0|^2 + \sum_{j\geq 1}|\nabla V^j|^2$, and the conditions defining $\mathcal{V}$ are those of [10, Remark 5]. The norm $|V|^2$ is not written out in [10]; we take $|V|^2 = \langle V, V\rangle_L$, the squared hyperbolic length of $V \in T_X\mathbb{H}^3$. With this choice the formula is the second variation of the energy, as stated in [10]; with the Euclidean norm of $\mathbb{R}^4$ the resulting form would exceed S_E by $4\int_\Sigma (V^0)^2\,\mathrm{d}A$ on $\mathcal{V}$. For completeness we check that S_E is the second variation of the energy $E = \frac{1}{2}\int_\Sigma \langle\nabla X, \nabla X\rangle_L\,\mathrm{d}A$, the conformal structure of Σ being fixed. Let X_t be a smooth variation with $X_t(\Sigma) \subset \mathbb{H}^3$, $X_t(\partial\Sigma) \subset \partial\mathbb{B}(r)$, $X_0 = X$ and $\partial_t X_t|_{t=0} = V$, and put $Z = \partial_t^2 X_t|_{t=0}$. Differentiating $\langle X_t, X_t\rangle_L = -1$, and $\langle X_t, e_0\rangle_L = -\cosh r$ on $\partial\Sigma$, twice gives $\langle Z, X\rangle_L = -\langle V, V\rangle_L$ and $\langle Z, e_0\rangle_L = 0$ on $\partial\Sigma$. Since $\Delta_\Sigma X = 2X$ (Section 4) and $\partial_\eta X = \eta$ is given by (9.1), Green's formula gives

$$\frac{\mathrm{d}^2E}{\mathrm{d}t^2}\Big|_{t=0} = \int_\Sigma \langle\nabla V, \nabla V\rangle_L\,\mathrm{d}A - \int_\Sigma \langle\Delta_\Sigma X, Z\rangle_L\,\mathrm{d}A + \int_{\partial\Sigma}\langle\partial_\eta X, Z\rangle_L\,\mathrm{d}\ell = S_E(V,V).$$

9.2. **Energy and area.**

Lemma 17. *Every $V \in \mathcal{V}$ can be written uniquely as $V = u\nu + dX(W)$, where $u \in C^\infty(\Sigma)$ and W is a smooth vector field on Σ tangent to $\partial\Sigma$ along $\partial\Sigma$; conversely, every such field lies in $\mathcal{V}$. For $V = u\nu + dX(W)$ and $V' = u'\nu + dX(W')$ in $\mathcal{V}$,*

$$S_E(V,V') = Q[u,u'] + \frac{1}{2}\int_\Sigma \langle 2u\,\mathrm{II} - (\mathcal{L}_W g)^\circ,\, 2u'\,\mathrm{II} - (\mathcal{L}_{W'}g)^\circ\rangle\,\mathrm{d}A, \tag{9.4}$$

where $Q[\cdot,\cdot]$ is the symmetric bilinear form associated with (3.1) *and $(\cdot)^\circ$ denotes the trace-free part of a symmetric 2-tensor.*

Proof. Since $\langle V, X\rangle_L = 0$, the field V is tangent to $\mathbb{H}^3$; put $u = \langle V, \nu\rangle_L$ and let $dX(W)$ be its tangential part. By (9.1), $e_0 = \cosh r\, X - \sinh r\,\eta$ along $\partial\Sigma$, so $\langle V, e_0\rangle_L = -\sinh r\, g(W,\eta)$ there. Thus the boundary condition in $\mathcal{V}$ is equivalent to $W \perp \eta$.

By polarization it suffices to take $V = V'$. By the Gauss formula,

$$D_{e_i}\nu = -dX(\mathcal{A}e_i), \qquad D_{e_i}(dX(W)) = dX(\nabla_{e_i}W) + \mathrm{II}(e_i, W)\,\nu + g(e_i, W)\,X.$$

Since X, ν and $dX(T\Sigma)$ are mutually orthogonal, with $\langle \nu, \nu\rangle_L = 1$ and $\langle X, X\rangle_L = -1$,

$$\langle \nabla(u\nu), \nabla(u\nu)\rangle_L = |\nabla u|^2 + |\mathrm{II}|^2 u^2,$$
$$\langle \nabla(dX(W)), \nabla(dX(W))\rangle_L = |\nabla W|^2 + |\mathcal{A}W|^2 - |W|^2,$$
$$\langle \nabla(u\nu), \nabla(dX(W))\rangle_L = \mathrm{II}(\nabla u, W) - u\langle \mathrm{II}, \nabla W\rangle.$$

(a) By (3.1), $S_E(u\nu, u\nu) = Q[u] + 2\int_\Sigma |\mathrm{II}|^2 u^2 \,\mathrm{d}A$.

(b) The shape operator has eigenvalues $\pm K/B^2$ (Section 2), so $|\mathcal{A}W|^2 = \frac{1}{2}|\mathrm{II}|^2|W|^2$, and the Gauss equation gives the Gauss curvature $\mathcal{K} = -1 - \frac{1}{2}|\mathrm{II}|^2$ of Σ. Hence

$$S_E(dX(W), dX(W)) = \int_\Sigma (|\nabla W|^2 - \mathcal{K}|W|^2)\,\mathrm{d}A - c\int_{\partial\Sigma} |W|^2 \,\mathrm{d}\ell.$$

On the other hand, for every vector field W on a surface,

$$\tfrac{1}{2}|(\mathcal{L}_W g)^\circ|^2 = |\nabla W|^2 - \mathcal{K}|W|^2 + \operatorname{div}(\nabla_W W - (\operatorname{div} W)\, W),$$

as follows from $\frac{1}{2}|(\mathcal{L}_W g)^\circ|^2 = \frac{1}{2}|\mathcal{L}_W g|^2 - (\operatorname{div} W)^2$ and the Ricci identity $\nabla_i\nabla_j W^i = \nabla_j \operatorname{div} W + \mathcal{K} W_j$. If $W = w\tau$ on $\partial\Sigma$, with τ a unit tangent of $\partial\Sigma$, then

$$g(\nabla_W W - (\operatorname{div} W)W, \eta) = w^2 g(\nabla_\tau \tau, \eta) = -w^2\langle \tau, \nabla^{\mathbb{H}^3}_\tau \eta\rangle_L = -c\, w^2,$$

because η is the outward unit normal of the supporting sphere (Section 3). Integrating, $S_E(dX(W), dX(W)) = \frac{1}{2}\int_\Sigma |(\mathcal{L}_W g)^\circ|^2 \,\mathrm{d}A$.

(c) As in the proof of Lemma 4, the Codazzi equation and minimality give $\operatorname{div}\mathcal{A} = 0$, hence $\operatorname{div}(\mathcal{A}W) = \langle \mathrm{II}, \nabla W\rangle$. On $\partial\Sigma$ one has $\mathrm{II}(\tau, \eta) = \langle \nu, D_\tau \eta\rangle_L = c\,\langle \nu, \tau\rangle_L = 0$, since the supporting sphere is umbilic. Integration by parts therefore gives $\int_\Sigma \mathrm{II}(\nabla u, W)\,\mathrm{d}A = -\int_\Sigma u\langle \mathrm{II}, \nabla W\rangle \,\mathrm{d}A$. The terms of order zero vanish because $\langle \nu, dX(W)\rangle_L = 0$, and so

$$S_E(u\nu, dX(W)) = -2\int_\Sigma u\langle \mathrm{II}, \nabla W\rangle \,\mathrm{d}A = -\int_\Sigma u\,\langle \mathrm{II}, (\mathcal{L}_W g)^\circ\rangle \,\mathrm{d}A,$$

where in the last step we used that II is symmetric and trace-free.

Adding (a), twice (c), and (b) gives (9.4) for $V = V'$. □

9.3. **Conical fields.**

Lemma 18. *Let C be a real 3×3 matrix, $P = (0, C\hat{X})$, $f_C = \langle P, X\rangle_L = \hat{X}\cdot C\hat{X}$ and $V_C = P + f_C X$. Then $\langle V_C, X\rangle_L = 0$. If $f_C = 0$ on $\partial\Sigma$, then $V_C \in \mathcal{V}$ and*

$$S_E(V_C, V_C) = -\int_\Sigma |\nabla f_C|^2 \,\mathrm{d}A.$$

Proof. We have $\langle V_C, X\rangle_L = f_C + f_C\langle X, X\rangle_L = 0$, and $\langle V_C, e_0\rangle_L = -f_C X^0$ vanishes on $\partial\Sigma$ when f_C does. By (9.2) and (9.3), $S_E(U, V) = -\mathcal{R}(U^0, V^0) + \sum_{j=1}^3 \mathcal{R}(U^j, V^j)$. For $j = 1, 2, 3$ one has $\Delta_\Sigma X^j = 2X^j$ in Σ, and $\partial_\eta X^j = cX^j$ on $\partial\Sigma$ by the spatial components of (9.1). By Green's formula, $\mathcal{R}(X^j, w) = 0$ for every $w \in H^1(\Sigma)$. Since $P^0 = 0$ and each P^j is a linear combination of X^1, X^2, X^3, it follows that $S_E(P, V) = 0$ for every V. Therefore $S_E(V_C, V_C) = S_E(f_C X, f_C X)$. Finally $D_{e_i}(f_C X) = (e_i f_C)X + f_C\, dX(e_i)$, so $\langle \nabla(f_C X), \nabla(f_C X)\rangle_L = -|\nabla f_C|^2 + 2f_C^2$, while $\langle f_C X, f_C X\rangle_L = -f_C^2$. Hence

$$S_E(f_C X, f_C X) = \int_\Sigma (-|\nabla f_C|^2 + 2f_C^2 - 2f_C^2)\,\mathrm{d}A + c\int_{\partial\Sigma} f_C^2 \,\mathrm{d}\ell = -\int_\Sigma |\nabla f_C|^2 \,\mathrm{d}A. \qquad \square$$

9.4. **Proof of Theorem 2.**

Proof of Theorem 2. Upper bound. Let $U \subset \mathcal{V}$ be a subspace on which S_E is negative definite. By Lemma 17, $S_E(V, V) \geq Q[\langle V, \nu\rangle_L]$ for $V \in \mathcal{V}$; moreover, if $\langle V, \nu\rangle_L = 0$, then $S_E(V, V) = \frac{1}{2}\int_\Sigma |(\mathcal{L}_W g)^\circ|^2 \,\mathrm{d}A \geq 0$. Hence the map $V \mapsto \langle V, \nu\rangle_L$ is injective on U, and Q is negative definite on its image. By Theorem 1, $\dim U \leq 4$.

Lower bound: the conical field. Let $\mu = B_0^2/g_1(S)^2$, which is well defined because $g_1(S) = A(S)\sinh\varphi(S) > 0$, and let $C = \operatorname{diag}(-\mu, 1, 1)$. By (2.3), $f_C = B^2 - \mu\, g_1^2$, a

function of s only. As B is even and g_1 is odd, f_C is even, and $f_C(\pm S) = 0$. Since $f_C(0) = B(0)^2 > 0$, the function f_C is not constant. Lemma 18 gives $V_C \in \mathcal{V}$ and

$$S_E(V_C, V_C) = -2\pi \int_{-S}^{S} f_C'(s)^2\, B(s)\, \mathrm{d}s < 0.$$

Moreover, by (2.3) and (2.7), $\langle V_C, \nu\rangle_L = \langle P, \nu\rangle_L = K - \mu\, g_1 \nu^1$, which depends only on s and is even.

Lower bound: three fields from Theorem 1. By Proposition 5, there is an odd function y_0 satisfying (3.3) with $J_0 y_0 = \lambda y_0$ and $\lambda < 0$. By Proposition 6, there is an even function y_1 satisfying (3.3) with $J_1 y_1 = \lambda' y_1$ and $\lambda' < 0$. Put $\zeta_1 = y_0(s)$, $\zeta_2 = y_1(s)\cos\theta$ and $\zeta_3 = y_1(s)\sin\theta$; then $Q[\zeta_i] < 0$. By (2.6) and the formulas for ν_s and ν_θ in Section 2, $\mathrm{II} = (K/B^2)\, ds^2 - K\, d\theta^2$. For $W = W^s \partial_s + W^\theta \partial_\theta$, the identity $(\mathcal{L}_W g)^\circ = 2u\,\mathrm{II}$ is equivalent to

$$B\,\partial_s\Big(\frac{W^s}{B}\Big) - \partial_\theta W^\theta = \frac{2Ku}{B^2}, \qquad \partial_\theta W^s + B^2\,\partial_s W^\theta = 0. \tag{9.5}$$

For ζ_1 take $W^\theta = 0$ and $W^s = B\gamma$, with

$$\gamma(s) = \int_{-S}^{s} \frac{2K\, y_0(\sigma)}{B(\sigma)^3}\, \mathrm{d}\sigma;$$

then $\gamma(\pm S) = 0$, because the integrand is odd. For ζ_2 take $W^s = B\alpha(s)\cos\theta$ and $W^\theta = \beta(s)\sin\theta$, where α solves

$$B\,(B\alpha')' - \alpha = B\Big(\frac{2Ky_1}{B^2}\Big)' \quad \text{on } (-S, S), \qquad \alpha(\pm S) = 0,$$

and $\beta = B\alpha' - 2Ky_1/B^2$. This boundary value problem is uniquely solvable: in the conformal coordinate t of Section 6 it reads $\ddot\alpha - \alpha = \frac{\mathrm{d}}{\mathrm{d}t}(2Ky_1/B^2)$, with Dirichlet conditions at $t = \pm T$. For ζ_3 take $W^s = B\alpha(s)\sin\theta$ and $W^\theta = -\beta(s)\cos\theta$. In each case (9.5) holds with $u = \zeta_i$, and $W^s(\pm S) = 0$, so the field W_i is tangent to $\partial\Sigma$. Put $V_i = \zeta_i \nu + dX(W_i) \in \mathcal{V}$. Since $2\zeta_i \mathrm{II} - (\mathcal{L}_{W_i} g)^\circ = 0$, Lemma 17 gives

$$S_E(V_i, V) = Q[\zeta_i, \langle V, \nu\rangle_L] \qquad \text{for every } V \in \mathcal{V}.$$

Orthogonality. The form Q is invariant under the rotations about the axis and under the reflection $s \mapsto -s$. Hence $Q[\zeta_i, \zeta_j] = 0$ for $i \neq j$, and $Q[\zeta_i, \langle V_C, \nu\rangle_L] = 0$ for $i = 1, 2, 3$, because $\langle V_C, \nu\rangle_L$ is radial and even, whereas ζ_1 is radial and odd and ζ_2, ζ_3 have angular frequency one. Thus the matrix of S_E on V_C, V_1, V_2, V_3 is diagonal with negative entries, and $\mathrm{ind}_E(\Sigma_a) \geq 4$. With the upper bound, $\mathrm{ind}_E(\Sigma_a) = 4$. Since $\mathrm{ind}_S(\Sigma_a) = 1$ [10, Theorem 18], we obtain $\mathrm{ind}_E(\Sigma_a) = 4 > 3 = 3\,\mathrm{ind}_S(\Sigma_a)$. □

Remark 19. The proof of Lemma 18 uses only $\Delta_\Sigma X = 2X$ and the Robin condition for the spatial coordinates. It therefore applies to every free boundary minimal surface in a geodesic ball of $\mathbb{H}^3$ whose boundary lies on a cone $\{\hat X \cdot C\hat X = 0\}$ and which is not contained in it.

The same computation in a geodesic ball of $\mathbb{S}^3_+$, with the Euclidean product and formula (8) in [10], gives $V_C = P - f_C X$ and $S_E(V_C, V_C) = +\int_\Sigma |\nabla f_C|^2\, \mathrm{d}A$. In the Euclidean unit ball one has $V_C = Cx$ and $S_E(V_C, \cdot) = 0$ on admissible fields. For the critical catenoid in the unit ball of $\mathbb{R}^3$, with the cone through its boundary circles, Cx is thus a null direction of the energy that is not induced by a rotation. Positive curvature moves this direction to the positive side, consistently with [10, Proposition 11]; negative curvature moves it to the negative side. In particular, the route of [10, Theorem 12], namely $\mathrm{ind} \leq n\,\mathrm{ind}_S + \dim \mathcal{M}$, where $\mathcal{M}$ is the moduli space of conformal structures on Σ (of dimension one for an annulus) and the inequality $\mathrm{ind} \leq \mathrm{ind}_E + \dim \mathcal{M}$ is due to V. Lima [8], cannot give $\mathrm{ind}(\Sigma_a) \leq 4$ in $\mathbb{H}^3$: by Theorem 2, already $\mathrm{ind}_E(\Sigma_a) = \mathrm{ind}(\Sigma_a) = 4$.

As in the proof of Lemma 18, $S_E(V, V) = -\mathcal{R}(V^0, V^0) + \sum_{j=1}^3 \mathcal{R}(V^j, V^j)$, and for $V \in \mathcal{V}$ the first term equals $-\int_\Sigma \big(|\nabla V^0|^2 + 2(V^0)^2\big)\, \mathrm{d}A \leq 0$, because $V^0 = 0$ on $\partial\Sigma$. In formula (8)

of [10] all components enter with the same sign, and the proof of [10, Proposition 11] bounds S_E from below by the spectral forms of the components. The opposite sign of the time component in $\mathbb{H}^3$ is the difficulty noted in [10, Remark 7].

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